Z_2 spin liquids in $S = 1/2$ Heisenberg model on kagome lattice: projective symmetry group study of Schwinger-fermion mean-field states

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Due to strong geometric frustration and quantum fluctuation, $S = 1/2$ quantum Heisenberg antiferromagnets on the kagome lattice has long been considered as an ideal platform to realize spin liquid (SL), a novel phase exhibiting fractionalized excitations without any symmetry breaking. A recent numerical study[1] of Heisenberg $S = 1/2$ kagome lattice model (HKLM) shows that in contrast to earlier results, the ground state is a singlet-gapped SL with signatures of Z_2 topological order. Motivated by this numerical discovery, we use projective symmetry group to classify all 20 possible Schwinger-fermion mean-field states of Z_2 SLs on kagome lattice. Among them we found only one gapped Z_2 SL (which we call $Z_2[0,\pi]\beta$ state) in the neighborhood of $U(1)$-Dirac SL state. Since its parent state, *i.e.* $U(1)$-Dirac SL is found[2] to be the lowest among many other candidate $U(1)$ SLs including the uniform resonating-valence-bond states, we propose this $Z_2[0,\pi]\beta$ state to be the numerically discovered SL ground state of HKLM.

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I. INTRODUCTION

At zero temperature all degrees of freedom tend to freeze and usually a variety of different orders, such as superconductivity and magnetism, will develop in different materials. However, in a quantum system with a large zero-point energy, one may expect a liquid-like ground state to exist even at $T = 0$. In a system consisting of localized quantum magnets, we call such a quantum-fluctuation-driven disordered ground state a quantum spin liquid (SL)[3]. It is an exotic phase with novel "fractionalized" excitations carrying only a fraction of the electron quantum number, *e.g.* spinons which carry spin but no charge. The internal structures of these SLs are so rich that they are beyond the description of Landau's symmetry breaking theory[4] of conventional ordered phases. Instead they are characterized by long-range quantum entanglement[5,6] encoded in the ground state, which is coined "topological order"[7,8] in contrast to the conventional symmetry-breaking order.

Geometric frustration in a system of quantum magnets would lead to a huge degeneracy of classical ground state configurations. The quantum tunneling among these classical ground states provides a mechanism to realize quantum SLs. The quest for quantum SLs in frustrated magnets (for a recent review see Ref. 9) has been pursued for decades. Among them the Heisenberg $S = 1/2$ kagome lattice model (HKLM)

$$H_{HKLM} = J \sum_{<i,j>} \boldsymbol{S}_i \cdot \boldsymbol{S}_j \tag{1}$$

has long been thought as a promising candidate. Here $< i, j >$ denotes i, j being a nearest neighbor pair. Experimental evidence of SL[10–13] has been observed in $ZnCu_3(OH)_6Cl_2$ (called herbertsmithite), a spin-half antiferromagnet on the kagome lattice. Theoretically, in lack of an exact solution of the two-dimensional (2D) quantum Hamiltonian (1) in the thermodynamic limit, in previous studies either a honeycomb valence bond crystal[14–18] (HVBC) with an enlarged 6×6-site unit cell, or a gapless SL[19] were proposed as the ground state of HKLM. However, recently an extensive density-matrix-renormalization-group (DMRG) study[1] on HKLM reveals the ground state of HKLM as a gapped SL, which substantially lowers the energy compared to HVBC. Besides, they also observe numerical signatures of Z_2 topological order in the SL state.

Motivated by this important numerical discovery, we try to find out the nature of this gapped Z_2 SL. Different Z_2 SLs on the kagome lattice have been previously studied using Schwinger-boson representation[20,21]. Here we propose the candidate states of symmetric Z_2 SLs on kagome lattice by Schwinger-fermion mean field approach[22–28]. Following is the summary of our results. First we use projective symmetry group[8] (PSG) to classify all 20 possible Schwinger-fermion mean-field ansatz of Z_2 SLs which preserve all the symmetry of HKLM, as shown in TABLE I. We analyze these 20 states and rule out some obviously unfavorable states: *e.g.* gapless states, and those states whose 1st nearest neighbor (n.n.) mean-field amplitudes must vanish due to symmetry. Then we focus on those Z_2 SLs in the neighborhood of the $U(1)$-Dirac SL[2]. In Ref. 2 it is shown that $U(1)$-Dirac SL has a significantly lower energy compared with other candidate $U(1)$ SL states, such as the uniform resonating-valence-bond (RVB) state(or the $U(1)$ SL-[0,0] state in notation of Ref. 2). We find out that there is only one gapped Z_2 SL, which we label as $Z_2[0,\pi]\beta$, in the neighborhood of (or continuously connected to) $U(1)$-Dirac SL. Therefore we propose this $Z_2[0,\pi]\beta$ state as a promising candidate state for the ground state of HKLM. The mean-field ansatz of $Z_2[0,\pi]\beta$ state is shown in FIG. 1(b). Our work also provides guideline for choosing variational

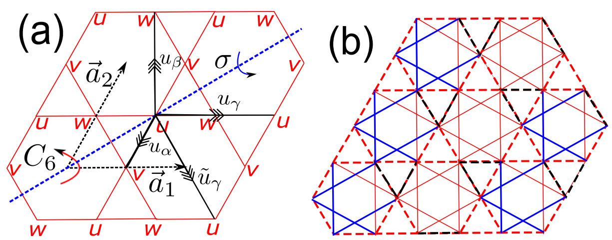

FIG. 1: (color online) (a) kagome lattice and the elements of its symmetry group. $\vec{a}_{1,2}$ are the translation unit vectors, C_6 denotes $\pi/3$ rotation around honeycomb center and $\boldsymbol{\sigma}$ represents mirror reflection along the dashed blue line. Here u_α and u_β denote 1st and 2nd nearest neighbor (n.n.) mean-field bonds while u_γ and $\tilde{u}_\gamma$ represent two kinds of independent 3rd n.n. mean-field bonds. (b) Mean-field ansatz of $Z_2[0,\pi]\beta$ state up to 2nd nearest neighbor. Colors in general denote the sign structure of mean-field bonds. Dashed lines denote 1st n.n. real hopping terms $\chi_1\sum_{<i,j>\alpha}(\nu_{ij}f^\dagger_{i\alpha}f_{j\alpha}+\ h.c.)$: red ones have $\nu_{ij}=1$ and black ones have $\nu_{ij}=-1$. Solid lines stand for 2nd n.n. hopping $\chi_2\sum_{<<ij>>\alpha}\nu_{ij}(f^\dagger_{i\alpha}f_{j\alpha}+\ h.c.)$ and singlet pairing $\sum_{<<ij>>\alpha\beta}\epsilon_{\alpha\beta}\nu_{ij}(\Delta_2 f^\dagger_{i\alpha}f^\dagger_{j\beta}+\ h.c.)$: again red ones have $\nu_{ij}=1$ and blue ones have $\nu_{ij}=-1$. Here $\chi_{1,2}$ and Δ_2 are real parameters after choosing a proper gauge.

states in future numeric studies of SL ground state on kagome lattice.

II. SCHWINGER-FERMION CONSTRUCTION OF SPIN LIQUIDS AND PROJECTIVE SYMMETRY GROUP (PSG)

A. Schwinger-fermion construction of symmetric spin liquids

In the Schwinger-fermion construction[22–27], we represent a spin-1/2 operator at site i by fermionic spinons $\{f_{i\alpha},\ \alpha=\uparrow,\downarrow\}$:

$$\vec{S}_i=\frac{1}{2}f^\dagger_{i\alpha}\vec{\sigma}_{\alpha\beta}f_{i\beta}. \tag{2}$$

Heisenberg hamiltonian $H=\sum_{<ij>}J_{ij}\vec{S}_i\cdot\vec{S}_j$ is represented as $H=\sum_{<ij>}-\frac{1}{2}J_{ij}\big(f^\dagger_{i\alpha}f_{j\alpha}f^\dagger_{j\beta}f_{j\beta}+\frac{1}{2}f^\dagger_{i\alpha}f_{i\alpha}f^\dagger_{j\beta}f_{j\beta}\big)$. This construction enlarges the Hilbert space of the original spin system. To obtain the physical spin state from a mean-field state of f-spinons, we need to enforce the following one-f-spinon-per-site constraint:

$$f^\dagger_{i\alpha}f_{i\alpha}=1, \qquad f_{i\alpha}f_{i\beta}\epsilon_{\alpha\beta}=0. \tag{3}$$

Mean-field parameters of symmetric SLs are $\Delta_{ij}\epsilon_{\alpha\beta}=-2\langle f_{i\alpha}f_{j\beta}\rangle$, $\chi_{ij}\delta_{\alpha\beta}=2\langle f^\dagger_{i\alpha}f_{j\beta}\rangle$, where $\epsilon_{\alpha\beta}$ is the completely antisymmetric tensor. Both terms are invariant under global $SU(2)$ spin rotations. After a Hubbard-Stratonovich transformation, the lagrangian of the spin system can be written as

$$\begin{aligned}L=&\sum_i\psi^\dagger_i\partial_\tau\psi_i+\sum_{<ij>}\frac{3}{8}J_{ij}\big[\frac{1}{2}\mathrm{Tr}(U^\dagger_{ij}U_{ij})\\&-(\psi^\dagger_iU_{ij}\psi_j+h.c.)\big]+\sum_i a^l_0(i)\psi^\dagger_i\tau^l\psi_i\end{aligned} \tag{4}$$

where two-component fermion notation $\psi_i\equiv(f_{i\uparrow},f^\dagger_{i\downarrow})$ is introduced for reasons that will be explained shortly. We use τ^0 to denote the 2×2 identity matrix and $\tau^{1,2,3}$ are the three Pauli matrices. U_{ij} is a matrix of mean-field amplitudes:

$$U_{ij}=\begin{pmatrix}\chi^\dagger_{ij} & \Delta_{ij}\\ \Delta^\dagger_{ij} & -\chi_{ij}\end{pmatrix}. \tag{5}$$

$a^l_0(i)$ are the local lagrangian multipliers that enforce the constraints Eq.(3).

In terms of ψ, Schwinger-fermion representation has an explicit $SU(2)$ gauge redundancy: a transformation $\psi_i\to W_i\psi_i$, $U_{ij}\to W_iU_{ij}W^\dagger_j$, $W_i\in SU(2)$ leaves the action invariant. This redundancy is originated from representation Eq.(2): this local $SU(2)$ transformation leaves the spin operators invariant and does not change physical Hilbert space. One can try to solve Eq.(4) by mean-field (or saddle-point) approximation. At mean-field level, U_{ij} and a^l_0 are treated as complex numbers, and a^l_0 must be chosen such that constraints (3) are satisfied at the mean field level: $\langle\psi^\dagger_i\tau^l\psi_i\rangle=0$. The mean-field ansatz can be written as:

$$H_{MF}=-\sum_{<ij>}\psi^\dagger_i\langle i|j\rangle\psi_j+\sum_i\psi^\dagger_ia^l_0\tau^l\psi_i. \tag{6}$$

where we defined $\langle i|j\rangle\equiv\frac{3}{8}J_{ij}U_{ij}$. Under a local $SU(2)$ gauge transformation $\langle i|j\rangle\to W_i\langle i|j\rangle W^\dagger_j$, but the physical spin state described by the mean-field ansatz $\{\langle i|j\rangle\}$ remains the same. By construction the mean-field ansatz does not break spin rotation symmetry, and the mean field solutions describe SL states if lattice symmetry is preserved. Different $\{\langle i|j\rangle\}$ ansatz may be in different SL phases. The mathematical language to classify different SL phases is projective symmetry group (PSG)[8].

B. Projective symmetry group (PSG) classification of topological orders in spin liquids

PSG characterizes the topological order in Schwinger-fermion representation: SLs described by different PSGs are different phases. It is defined as the collection of all combinations of symmetry group and $SU(2)$ gauge transformations that leave mean-field ansatz $\{\langle i|j\rangle\}$ invariant (as a^l_0 are determined self-consistently by $\{\langle i|j\rangle\}$, these transformations also leave a^l_0 invariant). The invariance

of a mean-field ansatz $\{\langle i|j\rangle\}$ under an element of PSG $G_U U$ can be written as

$$\begin{aligned} G_U U(\{\langle i|j\rangle\}) &= \{\langle i|j\rangle\}, \\ U(\{\langle i|j\rangle\}) &\equiv \{\langle \tilde{i|j}\rangle = \langle U^{-1}(i)|U^{-1}(j)\rangle\}, \\ G_U(\{\langle i|j\rangle\}) &\equiv \{\langle \tilde{i|j}\rangle = G_U(i)\langle i|j\rangle G_U(j)^\dagger\}, \\ G_U(i) &\in SU(2). \end{aligned} \tag{7}$$

Here $U \in SG$ is an element of symmetry group (SG) of the corresponding SL. In our case of symmetric SLs on the kagome lattice, we use (x, y, s) to label a site with sublattice index $s = u, v, w$ and $x, y \in \mathbb{Z}$. Bravais unit vector are chosen as $\vec{a}_1 = a\hat{x}$ and $\vec{a}_2 = \frac{a}{2}(\hat{x} + \sqrt{3}\hat{y})$ as shown in FIG. 1(a). The symmetry group is generated by time reversal operation $\boldsymbol{T}$, lattice translations $T_{1,2}$ along $\vec{a}_{1,2}$ directions, $\pi/3$ rotation C_6 around honeycomb plaquette center and the mirror reflection $\boldsymbol{\sigma}$ (for details see Appendix A). For example, if $U = T_1$ is the translation along $\vec{a}_1$-direction in Fig.1(a), $T_1(\{x, y, s\}) = \{x+1, y, s\}$. G_U is the gauge transformation associated with U such that $G_U U$ leave $\{\langle i|j\rangle\}$ invariant. Notice this condition (7) allows us to generate all symmetry-related mean-field bonds from one by the following relation:

$$\langle i|j\rangle = G_U(i)\langle U^{-1}(i)|U^{-1}(j)\rangle G_U^\dagger(j) \tag{8}$$

There is an important subgroup of PSG, the invariant gauge group (IGG), which is composed of all the pure gauge transformations in PSG: $IGG \equiv \{\{W_i\}|W_i\langle i|j\rangle W_j^\dagger = \langle i|j\rangle, W_i \in SU(2)\}$. In other words, $W_i = G_{\boldsymbol{e}}(i)$ is the pure gauge transformation associated with identity element $\boldsymbol{e} \in SG$ of the symmetry group. One can always choose a gauge in which the elements in IGG is site-independent. In this gauge, IGG can be the global Z_2 transformations: $\{G_{\boldsymbol{e}}(i) \equiv G_{\boldsymbol{e}} = \pm\tau^0\}$, the global $U(1)$ transformations: $\{G_{\boldsymbol{e}}(i) \equiv e^{\mathrm{i}\theta\tau^3}, \theta \in [0, 2\pi]\}$, or the global $SU(2)$ transformations: $\{G_{\boldsymbol{e}}(i) \equiv e^{\mathrm{i}\theta\hat{n}\cdot\vec{\tau}}, \theta \in (0, 2\pi], \hat{n} \in S^2\}$, and we term them as Z_2, $U(1)$ and $SU(2)$ state respectively.

The importance of IGG is that it controls the low energy gauge fluctuations of the corresponding SL states. Beyond mean-field level, fluctuations of $\langle i|j\rangle$ and a_0^l need to be considered and the mean-field state may or may not be stable. The low energy effective theory is described by fermionic spinon band structure coupled with a dynamical gauge field of IGG. For example, Z_2 state with gapped spinon dispersion can be a stable phase because the low energy Z_2 dynamical gauge field can be in the deconfined phase[29,30].

Notice that the condition $\{G_{\boldsymbol{e}}(i) \equiv G_{\boldsymbol{e}} = \pm\tau^0\}$ for a Z_2 SL leads to a series of consistent conditions for the gauge transformations $\{G_U(i)|U \in SG\}$, as shown in Appendix A. Gauge inequivalent solutions of these conditions (A4)-(A11) lead to different Z_2 SLs. Soon we will show that there are 20 Z_2 SLs on the kagome lattice that can be realized by a Schwinger-fermion mean-field ansatz $\{\langle i|j\rangle\}$.

III. Z_2 SPIN LIQUIDS ON THE KAGOME LATTICE AND $Z_2[0, \pi]\beta$ STATE

Following previous discussions, we use PSG to classify all possible 20 Z_2 SL states on kagome lattice in this section. As will be shown later, among them there is one gapped Z_2 SL labeled as $Z_2[0, \pi]\beta$ state in the neighborhood of $U(1)$-Dirac SL. This $Z_2[0, \pi]\beta$ SL state is the most promising candidate for the SL ground state of HKLM.

A. PSG classification of Z_2 spin liquids on kagome lattice

Applying the condition $G_{\boldsymbol{e}}(i) \equiv G_{\boldsymbol{e}} = \pm\tau^0$ to kagome lattice with symmetry group described in Appendix A, we obtain a series of consistent conditions for the gauge transformation $G_U(i)$, *i.e.* conditions (A4)-(A11). Solving these conditions we classify all the 20 different Schwinger-fermion mean-field states of Z_2 SLs on kagome lattice, as summarized in TABLE I. These 20 mean-field states correspond to different Z_2 SL phases, which cannot be continuously tuned into each other without a phase transition.

As discussed in Appendix B 2, from PSG elements $G_U(i)$ one can obtain all other symmetry-related mean-field bonds from one using symmetry condition (8). Therefore we use $u_\alpha \equiv \langle 0, 0, v|0, 0, u\rangle$ to represent 1st nearest neighbor (n.n.) mean-field bonds. $u_\beta \equiv \langle 0, 1, w|0, 0, u\rangle$ is the representative of 2nd n.n. mean-field bonds. There are two kinds of symmetry-unrelated 3rd n.n. mean-field bonds, represented by $u_\gamma = \langle 1, 0, u|0, 0, u\rangle$ and $\tilde{u}_\gamma = \langle 1, -1, u|0, 0, u\rangle$. The symmetry conditions for these mean-field bonds are summarized in (B13)-(B16). Besides, the on-site chemical potential terms $\Lambda(i)$ (which guarantee the physical constraint (3) on the mean-field level) also satisfy symmetry conditions (B12). We can show that $\Lambda(x, y, s) \equiv \Lambda_s$ for these 20 Z_2 SL states. The symmetry-allowed mean-field amplitudes/bonds are also summarized in TABLE I.

From TABLE I we can see there are 6 states, *i.e.* #7 − #12 that don't allow nonzero 1st n.n. mean-field amplitudes due to symmetry. Moreover, they cannot realize Z_2 SLs with up to 3rd n.n. mean-field amplitudes. Therefore they are unlikely to be the HKLM ground state. Ruling out these six Z_2 SLs, we can see the other 14 Z_2 SL states fall into 4 classes. To be specific, they are continuously connected to different parent $U(1)$ gapless SL states on kagome lattice. These parent $U(1)$ SL states in general have the following mean-field ansatz

$$H_{U(1)SL} = \chi_1 \sum_{<ij>} \nu_{ij}(f_{i\alpha}^\dagger f_{j\alpha} + \ h.c.) \tag{9}$$

where $\nu_{ij} = \pm 1$ characterizes the sign structure of hopping terms with $\chi_1 \in \mathbb{R}$. Different parent $U(1)$ SL

#	η_{12}	Λ_s	u_α	u_β	u_γ	$\tilde{u}_\gamma$	Label	Gapped?
1	$+1$	τ^2,τ^3	τ^2,τ^3	τ^2,τ^3	τ^2,τ^3	τ^2,τ^3	$Z_2[0,0]A$	Yes
2	-1	τ^2,τ^3	τ^2,τ^3	τ^2,τ^3	τ^2,τ^3	0	$Z_2[0,\pi]\beta$	**Yes**
3	$+1$	0	τ^2,τ^3	0	0	0	$Z_2[\pi,\pi]A$	No
4	-1	0	τ^2,τ^3	0	0	τ^2,τ^3	$Z_2[\pi,0]A$	No
5	$+1$	τ^3	τ^2,τ^3	τ^3	τ^3	τ^3	$Z_2[0,0]B$	Yes
6	-1	τ^3	τ^2,τ^3	τ^3	τ^3	τ^2	$Z_2[0,\pi]\alpha$	No
7	$+1$	0	0	τ^2,τ^3	0	0	-	-
8	-1	0	0	τ^2,τ^3	0	0	-	-
9	$+1$	0	0	0	τ^2,τ^3	0	-	-
10	-1	0	0	0	τ^2,τ^3	0	-	-
11	$+1$	0	0	τ^2	τ^2	0	-	-
12	-1	0	0	τ^2	τ^2	0	-	-
13	$+1$	τ^3	τ^3	τ^2,τ^3	τ^3	τ^3	$Z_2[0,0]D$	Yes
14	-1	τ^3	τ^3	τ^2,τ^3	τ^3	0	$Z_2[0,\pi]\gamma$	No
15	$+1$	τ^3	τ^3	τ^3	τ^2,τ^3	τ^3	$Z_2[0,0]C$	Yes
16	-1	τ^3	τ^3	τ^3	τ^2,τ^3	0	$Z_2[0,\pi]\delta$	No
17	$+1$	0	τ^2	τ^3	0	0	$Z_2[\pi,\pi]B$	No
18	-1	0	τ^2	τ^3	0	τ^3	$Z_2[\pi,0]B$	No
19	$+1$	0	τ^2	0	τ^2	0	$Z_2[\pi,\pi]C$	No
20	-1	0	τ^2	0	τ^2	τ^3	$Z_2[\pi,0]C$	No

TABLE I: Mean-field ansatz of 20 possible Z_2 SLs on a kagome lattice. In our notation of mean-field amplitudes $\langle x,y,s|0,0,u\rangle \equiv [x,y,s]$, this table summarizes all symmetry-allowed mean-field bonds up to 3rd n.n., *i.e.* 1st n.n. bond $u_\alpha = [0,0,v]$, 2nd n.n. bond $u_\beta = [0,1,w]$, 3rd n.n. bonds $u_\gamma = [1,0,u]$ and $\tilde{u}_\gamma = [1,-1,u]$ as shown in FIG. 1(a). Λ_s denote the on-site chemical potential terms which enforce the constraint (11). τ^0 is 2×2 identity matrix while $\tau^{1,2,3}$ are three Pauli matrices. $\tau^{0,3}$ denote hopping while $\tau^{1,2}$ denote pairing terms. 0 means the corresponding mean-field amplitudes must vanish due to symmetry. Red color denotes the shortest mean-field bonds necessary to realize a Z_2 SL. In other words, the mean-field amplitudes with red color break the $U(1)$ gauge redundancy down to Z_2 through Higgs mechanism. So in #3, #19 and #7 − #12 states a Z_2 SL cannot be realized with up to 3rd n.n. mean-field amplitudes. Note that #15 state needs only 3rd n.n. bond u_γ to realize a Z_2 SL ($\tilde{u}_\gamma$ not necessary) , while state #20 needs only $\tilde{u}_\gamma$ to realize a Z_2 SL (u_γ not necessary) . Notice that when $\eta_{12} = -1$ the mean-field ansatz (instead of the SL itself) will break translational symmetry and double the unit cell. There are six Z_2 SLs, *i.e.* #7 − #12 that don't allow any 1st n.n. mean-field bonds. Among the other 14 Z_2 SLs with nonvanishing 1st n.n. mean-field bonds, only five Z_2 SL states, *i.e.* #1, #2, #5, #13, #15 have gapped spinon spectra. #2 or $Z_2[0,\pi]\beta$ state in neighborhood of $U(1)$-Dirac SL is the most promising candidate of Z_2 SL for the HKLM ground state.

states are featured by the flux of f-spinon hopping phases around basic plaquette: honeycombs and triangles on the kagome lattice.

The simplest example is the so-called uniform RVB state with $\nu_{ij} \equiv +1$ for all 1st n.n. mean-field bonds. The hopping phase around any plaquette is $1 = \exp[\mathrm{i}0]$, and the corresponding flux is $[0,0]$ for [triangle,honeycomb] motifs. The 4 possible Z_2 spin liquids in the neighborhood[31] of uniform RVB states (*i.e.* $U(1)$ SL-$[0,0]$ state in Ref. 2) are classified in Appendix D. They

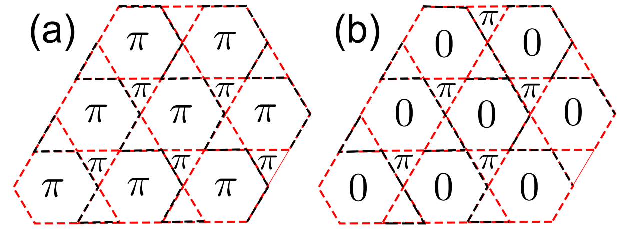

FIG. 2: (color online) Mean-field ansatz of (a) $U(1)$ SL-$[\pi,\pi]$ state and (b) $U(1)$ SL-$[\pi,0]$ state, with 1st n.n. real hopping terms $H_{MF} = \chi_1 \sum_{<ij>\alpha}(\nu_{ij} f^\dagger_{i\alpha} f_{j\alpha} + \ h.c.)$. Colors again denote the sign structure of mean-field bonds: red dashed lines have $\nu_{ij} = +1$ and black dashed lines have $\nu_{ij} = -1$.

are #1, #5, #15, #13 in TABLE I and TABLE II. We label them as $Z_2[0,0]A$, $Z_2[0,0]B$, $Z_2[0,0]C$ and $Z_2[0,0]D$ states. They all have gapped spectra of spinons.

The ansatz of two other parent $U(1)$ SLs are shown in FIG. 2. They both have π-flux piercing through a triangle basic plaquette. Following the above notations of hopping phase in [triangle,honeycomb] motifs, with either π-flux or 0-flux through the honeycomb plaquette, they are called $U(1)$ SL-$[\pi,\pi]$ state and $U(1)$ SL-$[\pi,0]$ state. There are three Z_2 SLs in the neighborhood of both $U(1)$ SL states, *i.e.* #3, #17, #19 around $U(1)$ SL-$[\pi,\pi]$ state and #4, #18, #20 around $U(1)$ SL-$[\pi,0]$ state. All these six Z_2 SLs have gapless spinon spectra, inherited from the two parent gapless $U(1)$ SLs. To be precise, the spinon band structure of these six Z_2 SL states are featured by a doubly-degenerate flat band and a Dirac cone at Brillouin-zone center. This is in contrast to the numerically observed gap in two-spinon spectrum[1], thus we can also rule out these 6 Z_2 SLs for the HKLM ground state.

Another $U(1)$ SL state is the so called $U(1)$-Dirac SL or $U(1)$ SL-$[0,\pi]$ state. Its mean-field ansatz is shown by the 1st n.n. bonds in FIG. 1(b). Clearly π-flux pierces through certain triangle plaquette with no flux through the honeycomb plaquette. According to variational Monte Carlo studies[2,32], this $U(1)$-Dirac SL have substantially lower energy compared to many other competing phases, including the uniform RVB state. Therefore we shall focus on those Z_2 SLs in the neighborhood of the $U(1)$-Dirac SL in our search of the HKLM ground state. We need to mention that although unlikely, the four Z_2 SLs in the neighborhood of uniform RVB state, or $U(1)$ SL-$[0,0]$ state are potentially possible to be the HKLM ground state.

In a previous study using PSG in Schwinger-boson representation[21], it was shown that there are 8 different Schwinger-boson mean-field ansatz of Z_2 SLs on the kagome lattice which preserve all lattice symmetry. However, these 8 Z_2 SLs may or may not preserve time-reversal symmetry. One can show that requiring all lattice symmetry *and time-reversal symmetry*, there are 16 different Schwinger-boson Z_2 SLs on the kagome lattice. The relation between the 20 Z_2 SLs in Schwinger-

fermion representation (see TABLE I) and the 16 Z_2 SLs in Schwinger-boson representation are not clear. To clarify the relation between SL states in these two different representations, one can compare the neighboring (ordered) phases of the SLs, *e.g.* by computing the vison quantum numbers[34] of SL states.

B. $Z_2[0,\pi]\beta$ state as a promising candidate for the HKLM ground state

How to find those Z_2 SLs in the neighborhood of (or continuously connected to) the $U(1)$-Dirac SL? Naively, we expect the mean-field ansatz of these Z_2 SLs can be obtained from that of $U(1)$-Dirac SL by adding an infinitesimal perturbation. To be specific, we require an infinitesimal spinon pairing term on top of the $U(1)$-Dirac SL mean-field ansatz (9) or (C1) to break the IGG from $U(1)$ to Z_2 through Higgs mechanism. Mathematically, we need to find those Z_2 SL states whose PSG is a subgroup of the $U(1)$-Dirac SL's PSG[31]. Such Z_2 SL states are defined to be in the neighborhood of $U(1)$-Dirac SL. Similar criterion applies to the neighboring Z_2 SL states of any parent $U(1)$ or $SU(2)$ SL state.

We find out all four Z_2 SLs in the neighborhood of $U(1)$-Dirac SLs in Appendix C. They are states #6, #2, #14, #16 in TABLE I, labeled as $Z_2[0,\pi]\alpha$, $Z_2[0,\pi]\beta$, $Z_2[0,\pi]\gamma$ and $Z_2[0,\pi]\delta$ states respectively. Since the effective theory of a $U(1)$-Dirac SL is an 8-component Dirac fermion coupled with dynamical $U(1)$ gauge field[2,33], we can find out all symmetry-allowed mass terms that can open up a gap in the Dirac-like spinon spectrum. Following detailed calculations in Appendix C, we can see that among the four Z_2 SLs around the $U(1)$-Dirac SL, only one state, *i.e.* $Z_2[0,\pi]\beta$ (state #2 in TABLE I and II) can generate a mass gap in the spinon spectrum. In other 3 states the Dirac cone in spinon spectrum is protected by symmetry. The mean-field ansatz of $Z_2[0,\pi]\beta$ SL state up to 2nd n.n. is shown in FIG. 1(b):

$$\begin{aligned} H_{MF} = \textstyle\sum_i(\lambda_3\sum_\alpha f^\dagger_{i\alpha}f_{i\alpha}+\lambda_1 f^\dagger_{i\uparrow}f^\dagger_{i\downarrow}+h.c.) \\ +\chi_1\textstyle\sum_{<ij>\alpha}\nu_{ij}(f^\dagger_{i\alpha}f_{j\alpha}+h.c.)+ \\ \textstyle\sum_{<<ij>>}\nu_{ij}(\chi_2\sum_\alpha f^\dagger_{i\alpha}f_{j\alpha}+\Delta_2\sum_{\alpha\beta}\epsilon^{\alpha\beta}f^\dagger_{i\alpha}f^\dagger_{j\beta}+h.c.) \end{aligned} \tag{10}$$

where $\epsilon^{\alpha\beta}$ is the completely anti-symmetric tensor. We only list up to 2nd n.n. mean-field amplitudes because as shown in TABLE I (see also Appendix C), this $Z_2[0,\pi]\beta$ state only needs 2nd n.n. pairing terms to realize a Z_2 SL. We can always choose a proper gauge so that mean-field parameters $\chi_{1,2}$ and Δ_2 are all real. The sign structure of $\nu_{ij}=\pm1$ are shown in FIG. 1(b), with red denoting $\nu_{ij}=+1$ and other colors representing $\nu_{ij}=-1$. As discussed in Appendix C, the 2nd n.n. singlet-pairing term $\Delta_2\neq0$ not only break the $U(1)$ gauge symmetry down to Z_2, but also opens up a mass gap in the spinon spectrum. The on-site chemical potential $\lambda_{1,3}$ are self-consitently determined by the following constraint:

$$\begin{aligned} \textstyle\sum_i\langle f^\dagger_{i\uparrow}f^\dagger_{i\downarrow}\rangle=\sum_i\langle f_{i\uparrow}f_{i\downarrow}\rangle=0, \\ \textstyle\sum_i(\sum_{\alpha=\uparrow,\downarrow}f^\dagger_{i\alpha}f_{i\alpha}-1)=0. \end{aligned} \tag{11}$$

For further n.n. mean-field ansatz see discussions in Appendix C.

IV. CONCLUSION

To summarize, motivated by the strong evidence of a Z_2 SL as the HKLM ground state in recent DMRG study[1], we classify all possible Z_2 SL states in Schwinger-fermion mean-field approach using PSG. We found 20 different Schwinger-fermion mean-field states of Z_2 SLs on kagome lattice, among which 6 states are unlikely due to vanishing 1st n.n. mean-field amplitude. In other 14 Z_2 SLs only 5 possess a gapped spinon spectrum, which is observed in the DMRG result[1]. These five symmetric Z_2 SL states are all in the neighborhood of certain parent $U(1)$ gapless SLs. To be precise, four are in the neighborhood of gapless uniform RVB (or $U(1)$ SL-$[0,0]$) state, while the other one, *i.e.* $Z_2[0,\pi]\beta$ is in the neighborhood of gapless $U(1)$-Dirac SL (or $U(1)$ SL-$[0,\pi]$) state. Previous variational Monte Carlo study[2] showed that gapless $U(1)$-Dirac SL has a substantially lower energy in comparison to the uniform RVB state. This suggests Z_2 SLs in the neighborhood of $U(1)$-Dirac SL should have lower energy compared to those in the neighborhood of uniform RVB state. Therefore we propose this $Z_2[0,\pi]\beta$ state with mean-field ansatz (10) shown in FIG. 1(b) as the HKLM ground state numerically detected in Ref. 1. Our work provides important insight for future numeric study, e.g. variational Monte Carlo study of Gutzwiller projected wavefunctions.

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Appendix A: Symmetry group of kagome lattice and algebra conditions for Z_2 spin liquids

As shown in FIG. 1(a), we label the three lattice sites in each unit cell with sublattice index $\{s=u,v,w\}$. Choosing Bravais unit vector as $\vec a_1=a\hat x$ and $\vec a_2=\frac{a}{2}(\hat x+\sqrt3\hat y)$, the positions of the three atoms in a unit cell labeled by indices $i=(x,y,s)$ are

$$\begin{aligned} \vec r(x,y,u)=(x+\tfrac12)\vec a_1+(y+\tfrac12)\vec a_2, \\ \vec r(x,y,v)=(x+\tfrac12)\vec a_1+y\vec a_2, \\ \vec r(x,y,w)=x\vec a_1+(y+\tfrac12)\vec a_2. \end{aligned} \tag{A1}$$

The symmetry group of such a two-dimensional kagome lattice is generated by the following operations

$$\begin{aligned} T_1: &\quad (x,y,s)\to(x+1,y,s); \\ T_2: &\quad (x,y,s)\to(x,y+1,s); \\ \boldsymbol{\sigma}: &\quad (x,y,u)\to(y,x,u), \\ &\quad (x,y,v)\to(y,x,w), \\ &\quad (x,y,w)\to(y,x,v); \\ C_6: &\quad (x,y,u)\to(-y-1,x+y+1,v), \\ &\quad (x,y,v)\to(-y,x+y,w), \\ &\quad (x,y,w)\to(-y-1,x+y,u). \end{aligned} \tag{A2}$$

together with time reversal $\boldsymbol{T}$.

The symmetry group of a kagome lattice is defined by the following algebraic relations between its generators:

$$\begin{aligned} \boldsymbol{T}^2 = \boldsymbol{\sigma}^2 = (C_6)^6 &= \boldsymbol{e}, \\ g^{-1}\boldsymbol{T}^{-1}g\boldsymbol{T} = \boldsymbol{e}, &\quad \forall\ g = T_{1,2}, \boldsymbol{\sigma}, C_6, \\ T_2^{-1}T_1^{-1}T_2T_1 &= \boldsymbol{e}, \\ \boldsymbol{\sigma}^{-1}T_1^{-1}\boldsymbol{\sigma}T_2 &= \boldsymbol{e}, \\ \boldsymbol{\sigma}^{-1}T_2^{-1}\boldsymbol{\sigma}T_1 &= \boldsymbol{e}, \\ {C_6}^{-1}T_2^{-1}C_6T_1 &= \boldsymbol{e}, \\ {C_6}^{-1}T_2^{-1}T_1C_6T_2 &= \boldsymbol{e}, \\ \boldsymbol{\sigma}^{-1}C_6\boldsymbol{\sigma}C_6 &= \boldsymbol{e}. \end{aligned} \tag{A3}$$

where $\boldsymbol{e}$ stands for the identity element in the symmetry group. Therefore the consistent conditions for a generic Z_2 PSGs on a kagome lattice is written as

$$[G_{\boldsymbol{T}}(i)]^2 = \eta_T\tau^0, \tag{A4}$$
$$G_{\boldsymbol{\sigma}}(\boldsymbol{\sigma}(i))G_{\boldsymbol{\sigma}}(i) = \eta_{\boldsymbol{\sigma}}\tau^0, \tag{A5}$$
$$G^\dagger_{T_1}(i)G^\dagger_{\boldsymbol{T}}(i)G_{T_1}(i)G_{\boldsymbol{T}}({T_1}^{-1}(i)) = \eta_{T_1\boldsymbol{T}}\tau^0, \tag{A6}$$
$$G^\dagger_{T_2}(i)G^\dagger_{\boldsymbol{T}}(i)G_{T_2}(i)G_{\boldsymbol{T}}({T_2}^{-1}(i)) = \eta_{T_2\boldsymbol{T}}\tau^0, \tag{A7}$$
$$G^\dagger_{\boldsymbol{\sigma}}(i)G^\dagger_{\boldsymbol{T}}(i)G_{\boldsymbol{\sigma}}(i)G_{\boldsymbol{T}}(\boldsymbol{\sigma}^{-1}(i)) = \eta_{\boldsymbol{\sigma}\boldsymbol{T}}\tau^0, \tag{A8}$$
$$G^\dagger_{C_6}(i)G^\dagger_{\boldsymbol{T}}(i)G_{C_6}(i)G_{\boldsymbol{T}}({C_6}^{-1}(i)) = \eta_{C_6\boldsymbol{T}}\tau^0, \tag{A9}$$
$$G^\dagger_{T_2}(T_1^{-1}(i))G^\dagger_{T_1}(i)G_{T_2}(i)G_{T_1}(T_2^{-1}(i)) = \eta_{12}\tau^0, \tag{A10}$$
$$\begin{aligned} G_{C_6}({C_6}^{-1}(i))G_{C_6}({C_6}^{-2}(i))G_{C_6}({C_6}^{3}(i))G_{C_6}({C_6}^{2}(i))\,\cdot \\ G_{C_6}({C_6}^{2}(i))G_{C_6}(C_6(i))G_{C_6}(i) = \eta_{C_6}\tau^0, \end{aligned} \tag{A11}$$
$$G^\dagger_{\boldsymbol{\sigma}}(T_2^{-1}(i))G^\dagger_{T_2}(i)G_{\boldsymbol{\sigma}}(i)G_{T_1}(\boldsymbol{\sigma}(i)) = \eta_{\boldsymbol{\sigma}T_1}\tau^0, \tag{A12}$$
$$G^\dagger_{\boldsymbol{\sigma}}(T_1^{-1}(i))G^\dagger_{T_1}(i)G_{\boldsymbol{\sigma}}(i)G_{T_2}(\boldsymbol{\sigma}(i)) = \eta_{\boldsymbol{\sigma}T_2}\tau^0, \tag{A13}$$
$$G^\dagger_{\boldsymbol{\sigma}}(C_6(i))G_{C_6}(C_6(i))G_{\boldsymbol{\sigma}}(i)G_{C_6}(\boldsymbol{\sigma}(i)) = \eta_{\boldsymbol{\sigma}C_6}\tau^0, \tag{A14}$$
$$G^\dagger_{C_6}(T_2^{-1}(i))G^\dagger_{T_2}(i)G_{C_6}(i)G_{T_1}({C_6}^{-1}(i)) = \eta_{C_6T_1}\tau^0, \tag{A15}$$
$$\begin{aligned} G^\dagger_{C_6}(T_2^{-1}T_1(i))G^\dagger_{T_2}(T_1(i))G_{T_1}(T_1(i))\,\cdot \\ G_{C_6}(i)G_{T_2}({C_6}^{-1}(i)) = \eta_{C_6T_2}\tau^0. \end{aligned} \tag{A16}$$

for any lattice site $i=(x,y,s)$. Here all ηs are Z_2 integers characterizing different SLs: different (gauge inequivalent) choices of these Z_2 integers (different Z_2 PSGs) correspond to different Z_2 SLs. Notice that under a local guage transformation $W(i)\in SU(2)$ the PSG element $G_U(i)$ transforms as

$$G_U(i)\to W(i)G_U(i)W^\dagger(U^{-1}(i)) \tag{A17}$$

Appendix B: Classification of all Z_2 spin liquids on kagome lattice

1. Classification of Z_2 algebraic PSGs on kagome lattice

In this section we classify all possible Z_2 spin liquids on a kagome lattice. Mathematically we need to find out all gauge-inequivalent solutions of algebraic conditions (A4)-(A15) for Z_2 PSGs.

First from condition (A10) we can always choose a proper gauge so that

$$G_{T_1}(x,y,s) = \eta_{12}^y\tau^0, \quad G_{T_2}(x,y,s)\equiv\tau^0. \tag{B1}$$

From (A12) and (A13) we can see $G_{\boldsymbol{\sigma}}(x,y,s) = \eta^y_{\boldsymbol{\sigma}T_1}\eta^x_{\boldsymbol{\sigma}T_2}\eta_{12}^{xy}g_{\boldsymbol{\sigma}}(s)$. Condition (A5) further determine $\eta_{\boldsymbol{\sigma}T_1}=\eta_{\boldsymbol{\sigma}T_2}$ and therefore we have

$$G_{\boldsymbol{\sigma}}(x,y,s) = \eta^{x+y}_{\boldsymbol{\sigma}T_1}\eta_{12}^{xy}g_{\boldsymbol{\sigma}}(s)$$

where $SU(2)$ matrices $g_{\boldsymbol{\sigma}}(s)$ satisfy

$$g_{\boldsymbol{\sigma}}(w)g_{\boldsymbol{\sigma}}(v) = \big[g_{\boldsymbol{\sigma}}(u)\big]^2 = \eta_{\boldsymbol{\sigma}}\tau^0 \tag{B2}$$

Notice that we can always choose a proper global Z_2 gauge on $G_{T_1}(x,y,s)$ (which doesn't change the mean-field ansatz) so that $\eta_{C_6T_2}=1$ in (A16). From (A15) and (A16) it's straightforward to show that $G_{C_6}(x,y,u/v) = \eta^{x+y}_{C_6T_1}\eta_{12}^{xy+x(x+1)/2}g_{C_6}(u/v)$ and $G_{C_6}(x,y,w) = \eta^{x+y}_{C_6T_1}\eta_{12}^{x+y+xy+x(x+1)/2}g_{C_6}(w)$. It's condition (A14) that determines $\eta_{C_6T_1}=\eta_{\boldsymbol{\sigma}T_1}\eta_{12}$ and finally we have

$$\begin{aligned} G_{C_6}(x,y,u/v) &= \eta^{x+y}_{\boldsymbol{\sigma}T_1}\eta_{12}^{xy+\frac{x(x+1)}{2}}g_{C_6}(u/v), \\ G_{C_6}(x,y,w) &= (\eta_{12}\eta_{\boldsymbol{\sigma}T_1})^{x+y}\eta_{12}^{xy+\frac{x(x+1)}{2}}g_{C_6}(w). \end{aligned}$$

where $SU(2)$ matrices $g_{C_6}(s)$ satisfy

$$\big[g_{C_6}(w)g_{C_6}(v)g_{C_6}(u)\big]^2 = \eta_{12}\eta_{C_6}\tau^0, \tag{B3}$$
$$\big[g_{\boldsymbol{\sigma}}(v)g_{C_6}(w)\big]^2 = g_{\boldsymbol{\sigma}}(w)g_{C_6}(v)g_{\boldsymbol{\sigma}}(u)g_{C_6}(u) = \eta_{\boldsymbol{\sigma}}\eta_{\boldsymbol{\sigma}C_6}\tau^0. \tag{B4}$$

according to (A11) and (A14).

Now through a gauge transformation $W(x,y,s) = \eta^y_{\boldsymbol{\sigma}T_1}$ we can fix $\eta_{\boldsymbol{\sigma}T_{1,2}}=1$ and the PSG elements become

$$G_{\boldsymbol{\sigma}}(x,y,s) = \eta_{12}^{xy}g_{\boldsymbol{\sigma}}(s); \tag{B5}$$
$$\begin{aligned} G_{C_6}(x,y,u/v) &= \eta_{12}^{xy+\frac{x(x+1)}{2}}g_{C_6}(u/v), \\ G_{C_6}(x,y,w) &= \eta_{12}^{xy+x+y+\frac{x(x+1)}{2}}g_{C_6}(w). \end{aligned} \tag{B6}$$

#	η_{12}	$g_{\boldsymbol{\sigma}}(u)$	$g_{\boldsymbol{\sigma}}(v)$	$g_{\boldsymbol{\sigma}}(w)$	$g_{C_6}(u)$	$g_{C_6}(v)$	$g_{C_6}(w)$	Label
1	$+1$	τ^0	τ^0	τ^0	τ^0	τ^0	τ^0	$Z_2[0,0]A$
2	-1	τ^0	τ^0	τ^0	τ^0	τ^0	τ^0	$Z_2[0,\pi]\beta$
3	$+1$	τ^0	τ^0	τ^0	τ^0	$-\tau^0$	$\mathrm{i}\tau^1$	$Z_2[\pi,\pi]A$
4	-1	τ^0	τ^0	τ^0	τ^0	$-\tau^0$	$\mathrm{i}\tau^1$	$Z_2[\pi,0]A$
5	$+1$	τ^0	τ^0	τ^0	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$Z_2[0,0]B$
6	-1	τ^0	τ^0	τ^0	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$Z_2[0,\pi]\alpha$
7	$+1$	$\mathrm{i}\tau^1$	τ^0	$-\tau^0$	τ^0	$\mathrm{i}\tau^1$	τ^0	-
8	-1	$\mathrm{i}\tau^1$	τ^0	$-\tau^0$	τ^0	$\mathrm{i}\tau^1$	τ^0	-
9	$+1$	$\mathrm{i}\tau^1$	τ^0	$-\tau^0$	τ^0	$-\mathrm{i}\tau^1$	$\mathrm{i}\tau^1$	-
10	-1	$\mathrm{i}\tau^1$	τ^0	$-\tau^0$	τ^0	$-\mathrm{i}\tau^1$	$\mathrm{i}\tau^1$	-
11	$+1$	$\mathrm{i}\tau^1$	τ^0	$-\tau^0$	$\mathrm{i}\tau^3$	$-\mathrm{i}\tau^2$	$\mathrm{i}\tau^3$	-
12	-1	$\mathrm{i}\tau^1$	τ^0	$-\tau^0$	$\mathrm{i}\tau^3$	$-\mathrm{i}\tau^2$	$\mathrm{i}\tau^3$	-
13	$+1$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$Z_2[0,0]D$
14	-1	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$Z_2[0,\pi]\gamma$
15	$+1$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	τ^0	τ^0	τ^0	$Z_2[0,0]C$
16	-1	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	τ^0	τ^0	τ^0	$Z_2[0,\pi]\delta$
17	$+1$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	τ^0	τ^0	$\mathrm{i}\tau^1$	$Z_2[\pi,\pi]B$
18	-1	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	τ^0	τ^0	$\mathrm{i}\tau^1$	$Z_2[\pi,0]B$
19	$+1$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$-\mathrm{i}\tau^3$	$\mathrm{i}\tau^2$	$Z_2[\pi,\pi]C$
20	-1	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$-\mathrm{i}\tau^3$	$\mathrm{i}\tau^2$	$Z_2[\pi,0]C$

TABLE II: A summary of all 20 gauge-inequivalent PSG's with $G_{\boldsymbol{T}}(x,y,s) = \mathrm{i}\tau^1$ on the kagome lattice. Notice that there is a free Z_2 integer $\eta_{12} = \pm 1$ in other PSG elements (B1), (B5) and (B6). They correspond to 20 different Z_2 spin liquids on the kagome lattice.

According to (A4), (A6) and (A7) we can see that $G_{\boldsymbol{T}}(x,y,s) = \eta_{T_1\boldsymbol{T}}^x \eta_{T_2\boldsymbol{T}}^y g_{\boldsymbol{T}}(s)$. (A9) and (A8) further determines $\eta_{T_1\boldsymbol{T}} = \eta_{T_2\boldsymbol{T}} = 1$ and by choosing a proper gauge we have

$$G_{\boldsymbol{T}}(x,y,s) = g_{\boldsymbol{T}}(s) \equiv \begin{cases} \tau^0, & \eta_{\boldsymbol{T}} = 1. \\ \mathrm{i}\tau^1, & \eta_{\boldsymbol{T}} = -1. \end{cases} \tag{B7}$$

which satisfy

$$\begin{aligned} g_{\boldsymbol{\sigma}}(u)g_{\boldsymbol{T}}(u) &= \eta_{\boldsymbol{\sigma}\boldsymbol{T}} g_{\boldsymbol{T}}(u) g_{\boldsymbol{\sigma}}(u), \\ g_{\boldsymbol{\sigma}}(v)g_{\boldsymbol{T}}(w) &= \eta_{\boldsymbol{\sigma}\boldsymbol{T}} g_{\boldsymbol{T}}(v) g_{\boldsymbol{\sigma}}(v), \\ g_{\boldsymbol{\sigma}}(w)g_{\boldsymbol{T}}(v) &= \eta_{\boldsymbol{\sigma}\boldsymbol{T}} g_{\boldsymbol{T}}(w) g_{\boldsymbol{\sigma}}(w); \end{aligned} \tag{B8}$$

$$\begin{aligned} g_{C_6}(u)g_{\boldsymbol{T}}(w) &= \eta_{C_6\boldsymbol{T}} g_{\boldsymbol{T}}(u) g_{C_6}(u), \\ g_{C_6}(v)g_{\boldsymbol{T}}(u) &= \eta_{C_6\boldsymbol{T}} g_{\boldsymbol{T}}(v) g_{C_6}(v), \\ g_{C_6}(w)g_{\boldsymbol{T}}(v) &= \eta_{C_6\boldsymbol{T}} g_{\boldsymbol{T}}(w) g_{C_6}(w). \end{aligned} \tag{B9}$$

according to (A9) and (A8).

In the following we find out all the gauge-inequivalent solutions of $SU(2)$ matrices $g_{\boldsymbol{T},\boldsymbol{\sigma},C_6}(s)$ satisfying the above conditions. They are summarized in TABLE .

(I) $g_{\boldsymbol{T}}(s) = \tau^0$ and therefore $\eta_{\boldsymbol{T}} = \eta_{\boldsymbol{\sigma}\boldsymbol{T}} = \eta_{C_6\boldsymbol{T}} = 1$:

Conditions (B8) and (B9) are automatically satisfied.

(i) $\eta_{\boldsymbol{\sigma}} = 1$:

Notice that under a global gauge transformation $W(x,y,s) \equiv W_s \in SU(2)$ the PSG elements transform as

$$\begin{aligned} g_{\boldsymbol{\sigma}}(u) &\to W_u g_{\boldsymbol{\sigma}}(u) W_u^\dagger, \\ g_{\boldsymbol{\sigma}}(v) &\to W_v g_{\boldsymbol{\sigma}}(v) W_w^\dagger, \\ g_{\boldsymbol{\sigma}}(w) &\to W_w g_{\boldsymbol{\sigma}}(w) W_v^\dagger; \\ g_{C_6}(u) &\to W_u g_{C_6}(u) W_w^\dagger, \\ g_{C_6}(v) &\to W_v g_{C_6}(v) W_u^\dagger, \\ g_{C_6}(w) &\to W_w g_{C_6}(w) W_v^\dagger. \end{aligned}$$

Thus from (B2) and (B4) we can always have $g_{\boldsymbol{\sigma}}(s) = \tau^0$ and $g_{C_6}(u) = \tau^0$, $g_{C_6}(v) = \eta_{\boldsymbol{\sigma}C_6}\tau^0$ by choosing a proper gauge.

(A) $\eta_{\boldsymbol{\sigma}C_6} = \eta_{12}\eta_{C_6} = 1$:
from (B3) we have $g_{C_6}(w) = \tau^0$.

(B) $\eta_{\boldsymbol{\sigma}C_6} = \eta_{12}\eta_{C_6} = -1$:
from (B3) we have $g_{C_6}(w) = \mathrm{i}\tau^3$ by gauge fixing.

(ii) $\eta_{\boldsymbol{\sigma}} = -1$:
from (B2) we have $g_{\boldsymbol{\sigma}}(v) = -g_{\boldsymbol{\sigma}}(w) = \tau^0$ and $g_{\boldsymbol{\sigma}}(u) = \mathrm{i}\tau^3$ by gauge fixing. Also from (B4) we can choose a gauge so that $g_{C_6}(u) = \tau^0$ and $g_{C_6}(v) = -\mathrm{i}\eta_{\boldsymbol{\sigma}C_6}\tau^3$.

(A) $\eta_{\boldsymbol{\sigma}C_6} = -1$:
In this case (B4) requires $g_{C_6}(w) = \tau^0$ and thus $\eta_{12}\eta_{C_6} = -1$ according to (B3).

(B) $\eta_{\boldsymbol{\sigma}C_6} = 1$:

(a) $\eta_{12}\eta_{C_6} = -1$:
Now from (B4) and (B3) we have $g_{C_6}(w) = \mathrm{i}\tau^1$ by gauge fixing.

(b) $\eta_{12}\eta_{C_6} = 1$:
by (B4) and (B3) we must have $g_{C_6}(w) = \mathrm{i}\tau^3$.

To summarize there are $2 \times (2+3) = 10$ different algebraic PSGs with $\eta_{\boldsymbol{T}} = 1$ and $g_{\boldsymbol{T}}(s) = \tau^0$.

(II) $g_{\boldsymbol{T}}(s) = \mathrm{i}\tau^1$ and $\eta_{\boldsymbol{T}} = -1$:

(i) $\eta_{\boldsymbol{\sigma}} = 1$:
According to (B2)and (B8), by choosing a proper gauge we can have $g_{\boldsymbol{\sigma}}(s) = \tau^0$ and $\eta_{\boldsymbol{\sigma}\boldsymbol{T}} = 1$. From (B3) and (B4) we also have $\left[g_{C_6}(w)\right]^2 = g_{C_6}(v)g_{C_6}(u) = \eta_{\boldsymbol{\sigma}C_6}\tau^0 = \eta_{12}\eta_{C_6}\tau^0$.

(A) $\eta_{12}\eta_{C_6} = \eta_{\boldsymbol{\sigma}C_6} = 1$:
From (B9), (B3) and (B4), by choosing gauge we have $g_{C_6}(s) = \tau^0$ and $\eta_{C_6\boldsymbol{T}} = 1$.

(B) $\eta_{12}\eta_{C_6} = \eta_{\boldsymbol{\sigma}C_6} = -1$:

(a) $\eta_{C_6\boldsymbol{T}} = 1$:
In this case we have $g_{C_6}(u) = -g_{C_6}(v) = \tau^0$ and $g_{C_6}(w) = \mathrm{i}\tau^1$ by choosing a proper gauge.

(b) $\eta_{C_6\boldsymbol{T}} = -1$:
In this case we can have $g_{C_6}(s) = \mathrm{i}\tau^3$ by choosing a proper gauge.

(ii) $\eta_{\boldsymbol{\sigma}} = -1$:

(A) $\eta_{\boldsymbol{\sigma}\boldsymbol{T}} = 1$:
From (B8) and (B2) we have $g_{\boldsymbol{\sigma}}(u) = \mathrm{i}\tau^1$ and $g_{\boldsymbol{\sigma}}(v) = -g_{\boldsymbol{\sigma}}(w) = \tau^0$ by proper gauge fixing. Also from (B4) we know $\left[g_{C_6}(w)\right]^2 = -\eta_{\boldsymbol{\sigma}C_6}\tau^0$ and $g_{C_6}(u)g_{C_6}(v) = -\mathrm{i}\eta_{\boldsymbol{\sigma}C_6}\tau^1$.

(a) $\eta_{\boldsymbol{\sigma}C_6} = -1$:

from (B9) and (B4), (B3) it's clear that $\eta_{C_6\boldsymbol{T}}=1$, $g_{C_6}(u)=g_{C_6}(w)=\tau^0$ and $g_{C_6}(v)=\mathrm{i}\tau^1$ through gauge fixing. Also we have $\eta_{12}\eta_{C_6}=-1$.

(b) $\eta_{\boldsymbol{\sigma}C_6}=1$:

(b1) $\eta_{C_6\boldsymbol{T}}=1$:

In this case $\eta_{12}\eta_{C_6}=1$, and we can always choose a proper gauge so that $g_{C_6}(u)=\tau^0$, $g_{C_6}(w)=-g_{C_6}(v)=\mathrm{i}\tau^1$.

(b2) $\eta_{C_6\boldsymbol{T}}=-1$:

In this case $\eta_{12}\eta_{C_6}=-1$, and we can always choose a proper gauge so that $g_{C_6}(v)=-\mathrm{i}\tau^2$, $g_{C_6}(u)=g_{C_6}(w)=\mathrm{i}\tau^3$.

(B) $\eta_{\boldsymbol{\sigma T}}=-1$:

Conditions (B8) and (B2) assert that $g_{\boldsymbol{\sigma}}(s)=\mathrm{i}\tau^3$ by proper gauge choosing.

(a) $\eta_{\boldsymbol{\sigma}C_6}=-1$:

In this case from (B4) we know $g_{C_6}(w)=\mathrm{i}\tau^3$, hence $\eta_{C_6\boldsymbol{T}}=-1$. Then we can always choose a gauge so that $g_{C_6}(u)=g_{C_6}(v)=\mathrm{i}\tau^3$ and so $\eta_{12}\eta_{C_6}=-1$ from (B3).

(b) $\eta_{\boldsymbol{\sigma}C_6}=1$:

(b1) $\eta_{C_6\boldsymbol{T}}=1$:

In this case from (B8),(B4) we have $g_{C_6}(u)=g_{C_6}(v)=\tau^0$ by a proper gauge choice. Meanwhile conditions (B3) and (B4) become $\left[g_{C_6}(w)\right]^2=\eta_{12}\eta_{C_6}\tau^0$ and $\left[\mathrm{i}\tau^3 g_{C_6}(w)\right]^2=-\tau^0$.

(b.1.1) $\eta_{12}\eta_{C_6}=1$:

here we have $g_{C_6}(w)=\tau^0$.

(b.1.2) $\eta_{12}\eta_{C_6}=-1$:

here we have $g_{C_6}(w)=\mathrm{i}\tau^1$.

(b2) $\eta_{C_6\boldsymbol{T}}=-1$:

In this case from (B8) and (B4) we can always choose a proper gauge so that $g_{C_6}(u)=-g_{C_6}(v)=\mathrm{i}\tau^3$. We also have $g_{C_6}(w)=\mathrm{i}\tau^2$ and $\eta_{12}\eta_{C_6}=-1$ from (B3).

To summarize there are $2\times(3+7)=20$ different algebraic PSGs with $\eta_{\boldsymbol{T}}=-1$ and $g_{\boldsymbol{T}}(s)=\mathrm{i}\tau^1$.

So in summary we have $10+20=30$ different Z_2 algebraic PSGs satisfying conditions (A4)-(A16). Among them there are at most 20 solutions that can be realized by a mean-field ansatz, since those PSGs with $g_{\boldsymbol{T}}(s)=\tau^0$ would require all mean-field bonds to vanish due to (B11). As a result there are 20 different Z_2 spin liquids on a kagome lattice.

2. Symmetry conditions on mean-field anstaz

Let's denote the mean-field bonds connecting sites $(0,0,u)$ and (x,y,s) as $[x,y,s]\equiv\langle x,y,s|0,0,u\rangle$. Using (8) we can generate any other mean-field bonds through symmetry operations (such as translations $G_{T_{1,2}}T_{1,2}$ and mirror reflection $G_{\boldsymbol{\sigma}}\boldsymbol{\sigma}$) from $[x,y,s]$. However these mean-field bonds cannot be chosen arbitrarily since they possess symmetry relation (8):

$$\langle i|j\rangle=G_U(i)\ \langle U^{-1}(i)|U^{-1}(j)\rangle\ G_U^\dagger(j) \tag{B10}$$

where U is any element in the symmetry group. Notice that for time reversal $\boldsymbol{T}$ we have

$$G_{\boldsymbol{T}}(i)\langle i|j\rangle G_{\boldsymbol{T}}^\dagger(j)=-\langle i|j\rangle \tag{B11}$$

We summarize these symmetry conditions on the mean-field bonds here:

(i) For $s=u$

$$\begin{aligned}\boldsymbol{T}:&\quad g_{\boldsymbol{T}}[x,y,u]g_{\boldsymbol{T}}^\dagger=-[x,y,u],\\ T_1^xT_2^{-x}\boldsymbol{\sigma}:&\quad [x,-x,u]\to[x,-x,u]^\dagger,\\ T_1^{x+1}T_2^{y+1}{C_6}^3:&\quad [x,y,u]\to[x,y,u]^\dagger,\\ \boldsymbol{\sigma}:&\quad [x,x,u]\to[x,x,u].\end{aligned}$$

(ii) For $s=v$

$$\begin{aligned}\boldsymbol{T}:&\quad g_{\boldsymbol{T}}[x,y,v]g_{\boldsymbol{T}}^\dagger=[x,y,v],\\ T_2^{y+1}\boldsymbol{\sigma}{C_6}^2:&\quad [0,y,v]\to[0,y,v]^\dagger,\\ T_1^{2-2y}T_2^{y-1}\boldsymbol{\sigma}{C_6}^{-1}:&\quad [1-2y,y,v]\to[1-2y,y,v]^\dagger.\end{aligned}$$

(iii) For $s=w$

$$\begin{aligned}\boldsymbol{T}:&\quad g_{\boldsymbol{T}}[x,y,w]g_{\boldsymbol{T}}^\dagger=[x,y,w],\\ T_1^{x-1}T_2^{2-2x}\boldsymbol{\sigma}C_6:&\quad [x,1-2x,w]\to[x,1-2x,w]^\dagger,\\ T_1^{x+1}\boldsymbol{\sigma}{C_6}^{-2}:&\quad [x,0,w]\to[x,0,w]^\dagger.\end{aligned}$$

Now let's consider several simplest examples. At first, on-site chemical potential terms $\Lambda(x,y,s)=\Lambda_s$ satisfy the following consistent conditions:

$$\begin{aligned}\tau^1\Lambda_s\tau^1&=-\Lambda_s;\\ g_{\boldsymbol{\sigma}}(u)\Lambda_u g_{\boldsymbol{\sigma}}^\dagger(u)&=\Lambda_u,\\ g_{\boldsymbol{\sigma}}(v)\Lambda_w g_{\boldsymbol{\sigma}}^\dagger(v)&=\Lambda_v,\\ g_{\boldsymbol{\sigma}}(w)\Lambda_v g_{\boldsymbol{\sigma}}^\dagger(w)&=\Lambda_w;\\ g_{C_6}(u)\Lambda_w g_{C_6}^\dagger(u)&=\Lambda_u,\\ g_{C_6}(v)\Lambda_u g_{C_6}^\dagger(v)&=\Lambda_v,\\ g_{C_6}(w)\Lambda_v g_{C_6}^\dagger(w)&=\Lambda_w.\end{aligned} \tag{B12}$$

In fact in all 20 Z_2 spin on a kagome lattice we all have $\Lambda_u=\Lambda_v=\Lambda_w\equiv\Lambda_s$ with a proper gauge choice.

All the 1st n.n. mean-field bonds can be generated from $u_\alpha\equiv[0,0,v]$. For a generic Z_2 spin liquid with PSG elements $G_{\boldsymbol{T}}(x,y,s)=\mathrm{i}\tau^1$ and (B1)(B5)(B6), the bond $u_\alpha=[0,0,v]$ satisfies the following consistent conditions:

$$\begin{aligned}\tau^1u_\alpha\tau^1&=-u_\alpha,\\ g_{\boldsymbol{\sigma}}(u)g_{C_6}(u)g_{C_6}(w)u_\alpha g_{C_6}^\dagger(v)g_{C_6}^\dagger(w)g_{\boldsymbol{\sigma}}^\dagger(v)&=u_\alpha^\dagger.\end{aligned} \tag{B13}$$

It follows immediately that for six Z_2 spin liquids, *i.e.* $\#7-12$ in TABLE II all n.n. mean-field bonds must vanish since $u_\alpha=0$ as required by (B13). Therefore it's unlikely that the Z_2 spin liquid realized in kagome Hubbard model would be one of these 6 states. In the following we study the rest 14 Z_2 spin liquids on the kagome lattice.

All 2nd n.n. mean-field bonds can be generated from $u_\beta \equiv [0,1,w]$ which satisfies the following symmetry conditions

$$\tau^1 u_\beta \tau^1 = -u_\beta, \tag{B14}$$
$$g_{\boldsymbol{\sigma}}(u) g_{C_6}(u) u_\beta g^\dagger_{C_6}(v) g^\dagger_{\boldsymbol{\sigma}}(w) = u^\dagger_\beta.$$

There are two kinds of 3rd n.n. mean-field bonds: the first kind can all be generated by $u_\gamma \equiv [1,0,u]$ which satisfies

$$\tau^1 u_\gamma \tau^1 = -u_\gamma, \tag{B15}$$
$$g_{C_6}(u) g_{C_6}(v) g_{C_6}(w) u_\gamma \big[g_{C_6}(u) g_{C_6}(v) g_{C_6}(w)\big]^\dagger = u^\dagger_\gamma.$$

the second kind can all be generated by $\tilde{u}_\gamma \equiv [1,-1,u]$ which satisfies

$$\tau^1 \tilde{u}_\gamma \tau^1 = -\tilde{u}_\gamma, \tag{B16}$$
$$g_{\boldsymbol{\sigma}}(u) \tilde{u}_\gamma g^\dagger_{\boldsymbol{\sigma}}(u) = \tilde{u}^\dagger_\gamma,$$
$$g_{C_6}(u) g_{C_6}(w) g_{C_6}(v) \tilde{u}_\gamma \big[g_{C_6}(u) g_{C_6}(w) g_{C_6}(v)\big]^\dagger = \eta_{12} \tilde{u}^\dagger_\gamma.$$

Appendix C: Z_2 spin liquids in the neighborhood of $U(1)$ SL-$[0,\pi]$ state

1. Mean-field ansatz of $U(1)$ SL-$[0,\pi]$ state

Following $SU(2)$ Schwinger fermion formulation with $\psi_i \equiv (f_{i\uparrow}, f^\dagger_{i\downarrow})^T$, we focus on those Z_2 spin liquids (SLs) in the neighborhood of $U(1)$ SL-$[0,\pi]$ state with the following mean-field ansatz:

$$\langle x,y,u|x,y,v\rangle = -\langle x,y,u|x,y,w\rangle = (-1)^x \chi \tau^3, \tag{C1}$$
$$\langle x+1,y,w|x,y,u\rangle = \langle x,y+1,v|x,y,u\rangle = -\langle x,y,v|x,y,w\rangle$$
$$= \langle x+1,y-1,w|x,y,v\rangle = \chi\tau^3.$$

where χ is a real hopping parameter. We define mean-field bonds $\langle x,y,s|x',y',s'\rangle$ in the following way

$$H_{MF} = \sum_{i,j} \psi^\dagger_i \langle i|j\rangle \psi_j + \ h.c. \tag{C2}$$

For convenience of later calculation we implement the following gauge transformation

$$\psi_{x,y,u} \to \mathrm{i}\,\tau^3 \psi_{x,y,u} \tag{C3}$$

and the original mean-field ansatz (C1) transforms to be

$$\langle x,y,u|x,y,v\rangle = -\langle x,y,u|x,y,w\rangle = \mathrm{i}\,(-1)^x \chi \tau^0, \tag{C4}$$
$$\langle x+1,y,w|x,y,u\rangle = \langle x,y+1,v|x,y,u\rangle = -\mathrm{i}\,\chi\tau^0,$$
$$-\langle x,y,v|x,y,w\rangle = \langle x+1,y-1,w|x,y,v\rangle = \chi\tau^3.$$

The projected symmetry group (PSG) corresponds to the above mean-field ansatz (C4) is

$$G_{\boldsymbol{T}}(x,y,v) = G_{\boldsymbol{T}}(x,y,w) = -G_{\boldsymbol{T}}(x,y,u) = g_{\boldsymbol{T}},$$
$$g_{\boldsymbol{T}} \tau^3 g^\dagger_{\boldsymbol{T}} = -\tau^3;$$
$$G_{T_2}(x,y,s) = g_{T_2}, \quad g_{T_2}\tau^3 g^\dagger_{T_2} = \tau_3;$$
$$G_{T_1}(x,y,v) = G_{T_1}(x,y,w) = -G_{T_1}(x,y,u)$$
$$= (-1)^{x+y} g_{T_1}, \quad g_{T_1}\tau^3 g^\dagger_{T_1} = \tau_3;$$
$$G_{\boldsymbol{\sigma}}(x,y,v) = G_{\boldsymbol{\sigma}}(x,y,w) = (-1)^{x+y+1} G_{\boldsymbol{\sigma}}(x,y,u)$$
$$= (-1)^{(x+y)(x+y+1)/2} g_{\boldsymbol{\sigma}}, \quad g_{\boldsymbol{\sigma}}\tau^3 g^\dagger_{\boldsymbol{\sigma}} = \tau^3;$$
$$G_{C_6}(x,y,u) = (-1)^{\frac{x(x+1)+y(y-1)}{2}} g_{C_6},$$
$$G_{C_6}(x,y,v) = -(-1)^{\frac{x(x-1)+y(y-1)}{2}} g_{C_6},$$
$$G_{C_6}(x,y,w) = \mathrm{i}\,(-1)^{\frac{x(x-1)+y(y-1)}{2}} g_{C_6}\tau^3,$$
$$g_{C_6}\tau^3 g^\dagger_{C_6} = \tau^3. \tag{C5}$$

so that the mean-field ansatz satisfy (8).

2. Classification of Z_2 spin liquids around $U(1)$ SL-$[0,\pi]$ state

Plugging (C5) into algebraic consistent conditions (A4)-(A15) yields four algebraic solutions of Z_2 PSGs around the $U(1)$ SL-$[0,\pi]$ state. Choosing a proper gauge they all satisfy

$$g_{\boldsymbol{T}} = \mathrm{i}\,\tau_1, \quad g_{T_1} = g_{T_2} = \tau^0, \tag{C6}$$
$$\eta_{\boldsymbol{T}} = \eta_{12} = \eta_{C_6 T_1} = -1,$$
$$\eta_{T_{1,2}\boldsymbol{T}} = \eta_{\boldsymbol{\sigma} T_{1,2}} = \eta_{C_6 T_2} = 1.$$

The four Z_2 PSGs near the $U(1)$ SL-$[0,\pi]$ state are featured by

$$(\#6)\ Z_2[0,\pi]\alpha: \quad g_{\boldsymbol{\sigma}} = g_{C_6} = \tau^0, \tag{C7}$$
$$\eta_{\boldsymbol{\sigma}} = \eta_{\boldsymbol{\sigma}\boldsymbol{T}} = 1,$$
$$\eta_{\boldsymbol{\sigma} C_6} = \eta_{C_6\boldsymbol{T}} = -\eta_{C_6} = -1;$$

$$(\#2)\ Z_2[0,\pi]\beta: \quad g_{\boldsymbol{\sigma}} = \tau^0,\ g_{C_6} = \mathrm{i}\,\tau^3, \tag{C8}$$
$$\eta_{\boldsymbol{\sigma}} = \eta_{\boldsymbol{\sigma}\boldsymbol{T}} = 1,$$
$$\eta_{\boldsymbol{\sigma} C_6} = \eta_{C_6\boldsymbol{T}} = -\eta_{C_6} = 1;$$

$$(\#14)\ Z_2[0,\pi]\gamma: \quad g_{\boldsymbol{\sigma}} = \mathrm{i}\,\tau^3,\ g_{C_6} = \tau^0, \tag{C9}$$
$$\eta_{\boldsymbol{\sigma}} = \eta_{\boldsymbol{\sigma}\boldsymbol{T}} = -1,$$
$$\eta_{\boldsymbol{\sigma} C_6} = \eta_{C_6\boldsymbol{T}} = -\eta_{C_6} = -1;$$

$$(\#16)\ Z_2[0,\pi]\delta: \quad g_{\boldsymbol{\sigma}} = g_{C_6} = \mathrm{i}\,\tau^3 \tag{C10}$$
$$\eta_{\boldsymbol{\sigma}} = \eta_{\boldsymbol{\sigma}\boldsymbol{T}} = -1,$$
$$\eta_{\boldsymbol{\sigma} C_6} = \eta_{C_6\boldsymbol{T}} = -\eta_{C_6} = 1.$$

Of course they belong to the 20 Z_2 spin liquids summarized in TABLE II.

3. Four possible Z_2 spin liquids around $U(1)$ SL-$[0,\pi]$ state: mean-field ansatz

a. *Consistent conditions on mean-field bonds*

Implementing the generic conditions mentioned earlier on several near neighbor mean-field bonds with PSG (C6)-(C10), we obtain the following consistent conditions:

(0) For on-site chemical potential terms $\Lambda_s(x,y,s) = \vec{\lambda}(x,y,s)\cdot\vec{\tau}$, translations operations $G_{T_{1,2}}T_{1,2}$ in PSG guarantee that $\Lambda_s(x,y,s) = \lambda_s(0,0,s) \equiv \Lambda_s,\ s = u,v,w$. They satisfy

$$\begin{aligned} g_{\boldsymbol{T}}\Lambda_s g_{\boldsymbol{T}}^\dagger &= -\Lambda_s; \qquad (C11)\\ g_{\boldsymbol{\sigma}}\Lambda_u g_{\boldsymbol{\sigma}}^\dagger = \Lambda_u,\ g_{\boldsymbol{\sigma}}\Lambda_v g_{\boldsymbol{\sigma}}^\dagger &= \Lambda_w,\ g_{\boldsymbol{\sigma}}\Lambda_w g_{\boldsymbol{\sigma}}^\dagger = \Lambda_v;\\ g_{C_6}\Lambda_u g_{C_6}^\dagger = \Lambda_v,\ (g_{C_6}\tau^3)&\Lambda_v(g_{C_6}\tau^3)^\dagger = \Lambda_w,\\ g_{C_6}\Lambda_w g_{C_6}^\dagger &= \Lambda_u. \end{aligned}$$

(I) For 1st neighbor mean-field bond $u_a \equiv [0,0,v]^\dagger$ (there is only one independent mean-field bond, meaning all other 1st neighbor bonds can be generated from [0,0,v] through symmetry operations)

$$\begin{aligned} g_{\boldsymbol{T}}u_a^\dagger g_{\boldsymbol{T}}^\dagger &= u_a^\dagger, \qquad (C12)\\ (g_{\boldsymbol{\sigma}}g_{C_6}^2\tau^3)u_a^\dagger(g_{\boldsymbol{\sigma}}g_{C_6}^2\tau^3)^\dagger &= -u_a. \end{aligned}$$

(II) For 2nd neighbor mean-field bond $u_b \equiv [0,1,w]$ we have

$$\begin{aligned} g_{\boldsymbol{T}}u_b g_{\boldsymbol{T}}^\dagger &= u_b, \qquad (C13)\\ (g_{\boldsymbol{\sigma}}g_{C_6})u_b(g_{\boldsymbol{\sigma}}g_{C_6})^\dagger &= -u_b^\dagger. \end{aligned}$$

(II) For 3rd neighbor mean-field bonds $u_{c1} \equiv [1,0,u]$ and $u_{c2} \equiv [1,-1,u]$ we have

$$\begin{aligned} g_{\boldsymbol{T}}u_{c1}g_{\boldsymbol{T}}^\dagger &= -u_{c1}, \qquad (C14)\\ (g_{C_6}^3\tau^3)u_{c1}(g_{C_6}^3\tau^3)^\dagger &= u_{c1}^\dagger. \end{aligned}$$

and

$$\begin{aligned} g_{\boldsymbol{T}}u_{c2}g_{\boldsymbol{T}}^\dagger &= -u_{c2}, \qquad (C15)\\ g_{\boldsymbol{\sigma}}u_{c2}g_{\boldsymbol{\sigma}}^\dagger &= u_{c2}^\dagger,\\ (g_{C_6}^3\tau^3)u_{c2}(g_{C_6}^3\tau^3)^\dagger &= -u_{c2}^\dagger. \end{aligned}$$

b. *Mean-field ansatz of the four Z_2 spin liquids near $U(1)$ SL-$[0,\pi]$ state*

For $Z_2[0,\pi]\alpha$ state with $g_{\boldsymbol{\sigma}} = g_{C_6} = \tau^0$ the mean-field ansatz are (up to 3rd neighbor mean-field bonds)

$$\begin{aligned} u_a = \mathrm{i}a_0\tau^0 + a_1\tau^1,\quad u_b &= \mathrm{i}b_0\tau^0, \qquad (C16)\\ u_{c1} = c_3\tau^3,\quad u_{c2} &= c_2\tau^2,\\ \Lambda_s = \lambda_3\tau^3,\quad s &= u,v,w. \end{aligned}$$

Since we are considering a phase perturbed from the $U(1)$ SL-$[0,\pi]$ state, we shall always assume $a_0 \neq 0$ (1st neighbor hopping terms) in the following discussion. A $Z_2[0,\pi]\alpha$ spin liquid can be realized by 1st neighbor mean-field singlet pairing terms with $a_1 \neq 0$.

For $Z_2[0,\pi]\beta$ state with $g_{\boldsymbol{\sigma}} = \tau^0,\ g_{C_6} = \mathrm{i}\tau^3$ the mean-field ansatz are (up to 3rd neighbor mean-field bonds)

$$\begin{aligned} u_a = \mathrm{i}a_0\tau^0 + a_1\tau^1,\quad u_b &= \mathrm{i}b_0\tau^0 + b_1\tau^1, \qquad (C17)\\ u_{c1} = c_2\tau^2 + c_3\tau^3,&\quad u_{c2} = 0,\\ \Lambda_u = \lambda_2\tau^2 + \lambda_3\tau^3,\ \Lambda_{v,w} &= -\lambda_2\tau^2 + \lambda_3\tau^3. \end{aligned}$$

A $Z_2[0,\pi]\beta$ spin liquid can be realized by 2nd neighbor pairing terms with $a_0b_1 - a_1b_0 \neq 0$.

For $Z_2[0,\pi]\gamma$ state with $g_{\boldsymbol{\sigma}} = \mathrm{i}\tau^3,\ g_{C_6} = \tau^0$ the mean-field ansatz are (up to 3rd neighbor mean-field bonds)

$$\begin{aligned} u_a = \mathrm{i}a_0\tau^0,\quad u_b &= \mathrm{i}b_0\tau^0 + b_1\tau^1, \qquad (C18)\\ u_{c1} = c_3\tau^3,&\quad u_{c2} = 0,\\ \Lambda_s = \lambda_3\tau^3,&\quad s = u,v,w. \end{aligned}$$

A $Z_2[0,\pi]\gamma$ spin liquid can be realized by 2nd neighbor pairing terms with $b_1 \neq 0$.

For $Z_2[0,\pi]\delta$ state with $g_{\boldsymbol{\sigma}} = g_{C_6} = \mathrm{i}\tau^3$ the mean-field ansatz are (up to 3rd neighbor mean-field bonds)

$$\begin{aligned} u_a = \mathrm{i}a_0\tau^0,&\quad u_b = \mathrm{i}b_0\tau^0, \qquad (C19)\\ u_{c1} = c_2\tau^2 + c_3\tau^3,&\quad u_{c2} = 0,\\ \Lambda_s = \lambda_3\tau^3,&\quad s = u,v,w. \end{aligned}$$

A $Z_2[0,\pi]\delta$ spin liquid can be realized by 3rd neighbor pairing terms with $c_2 \neq 0$.

4. Low-energy effective theory

The reciprocal unit vectors (corresponding to unit vectors $\vec{a}_{1,2}$) on a kagome lattice are $\vec{b}_1 = \frac{1}{a}(\hat{x} - \frac{1}{\sqrt{3}}\hat{y})$ and $\vec{b}_2 = \frac{1}{a}\frac{2}{\sqrt{3}}\hat{y}$, satisfying $\vec{a}_i\cdot\vec{b}_j = \delta_{i,j}$. In the mean-field ansatz (C4) of $U(1)$ SL-$[0,\pi]$ the unit cell is doubled whose translation unit vectors are $\vec{A}_1 = 2\vec{a}_1$ and $\vec{A}_2 = \vec{a}_2$. Accordingly the 1st BZ for such a mean-field ansatz is only half of the original 1st BZ with new reciprocal unit vectors being $\vec{B}_1 = \vec{b}_1/2$ and $\vec{B}_2 = \vec{b}_2$. Denoting the momentum as $\mathbf{k} \equiv (k_x,k_y)/a = k_1\vec{B}_1 + k_2\vec{B}_2$ with $|k_{1,2}| \leq \pi$, we have

$$k_1 = 2k_x,\quad k_2 = (k_x + \sqrt{3}k_y)/2. \qquad (C20)$$

The two Dirac cones in the spectra of $U(1)$ SL-$[0,\pi]$ state (C4) are located at $\pm\mathbf{Q}$ with

$$\mathbf{Q} = (0,\frac{\pi}{\sqrt{3}}) = \frac{\pi}{2}\vec{B}_2 \qquad (C21)$$

with the proper chemical potential $\Lambda(i) = \langle i|i\rangle = \chi(\sqrt{3}-1)\tau^3$ added to mean-field ansatz (C4).

For convenience we choose the following basis for Dirac-like Hamiltonian obtained from expansion around $\pm\mathbf{Q}$:

$$\begin{aligned}\phi_{+,\uparrow,A} &= \tfrac{1}{\sqrt{6}}e^{-\mathrm{i}\frac{1}{24}\pi}\cdot\\ &(e^{-\mathrm{i}\frac{11}{12}\pi},0,e^{\mathrm{i}\frac{11}{12}\pi},0,0,0,e^{-\mathrm{i}\frac{11}{12}\pi},0,e^{\mathrm{i}\frac{5}{12}\pi},0,\sqrt{2},0)^T,\\ \phi_{+,\uparrow,B} &= \tfrac{1}{\sqrt{6}}e^{-\mathrm{i}\frac{1}{24}\pi}\cdot\\ &(1,0,e^{-\mathrm{i}\frac{4}{3}\pi},0,\sqrt{2}e^{-\mathrm{i}\frac{11}{12}\pi},0,-1,0,e^{-\mathrm{i}\frac{5}{6}\pi},0,0,0)^T,\\ \phi_{-,\uparrow,b} &= R_{T_1}(k_1=0,k_2=-\tfrac{\pi}{2})\phi_{+,\uparrow,b},\\ \phi_{\pm,\downarrow,b} &= R_{\boldsymbol{T}}\phi_{\pm,\uparrow,b}.\end{aligned} \tag{C22}$$

where $\pm$ are valley index for two Dirac cones at $\pm\mathbf{Q}$ with Pauli matrices $\boldsymbol{\mu}$ and $b=A,B$ are band indices (for the two bands forming the Dirac cone) with Pauli matrices $\boldsymbol{\nu}$. Pseudospin indices $\Sigma=\uparrow,\downarrow$ are assigned to the two degenerate bands related by time reversal, with Pauli matrices $\boldsymbol{\sigma}$. The corresponding creation operators for these modes are $\Psi^\dagger_{\pm,\Sigma,b}=\psi^\dagger_{\pm\mathbf{Q}}\phi_{\pm,\Sigma,b}$ in the order of $(0,0,u),(0,0,v),(0,0,w),(1,0,u),(1,0,v),(1,0,w)$ for the six sites per doubled new unit cell. Notice that in terms of f-spinons we have $\psi^\dagger=(f^\dagger_\uparrow,f_\downarrow)$.

Here $R_{\boldsymbol{T}}\equiv I_{2\times2}\otimes\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\otimes g_{\boldsymbol{T}}$, $R_{T_2}(\mathbf{k})=e^{-\mathrm{i}k_2}I_{6\times6}\otimes g_{T_2}$ and $R_{T_1}(\mathbf{k})=\begin{bmatrix}0&-e^{-\mathrm{i}k_1}\\1&0\end{bmatrix}\otimes\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\otimes g_{T_1}$ are transformation matrices on 12-component eigenvectors for time reversal $\boldsymbol{T}$ and translation $T_{1,2}$ operations. By definition of PSG the eigenvectors $\phi_{\mathbf{k}}$ with momentum $\mathbf{k}=k_1\vec{B}_1+k_2\vec{B}_2\equiv(k_1,k_2)$ and energy E have the following symmetric properties:

$$\begin{aligned}\boldsymbol{T}:\quad &\tilde{\phi}_{(k_1,k_2)}(E)=R_{\boldsymbol{T}}\phi_{(k_1,k_2)}(-E),\\ T_1:\quad &\tilde{\phi}_{(k_1,k_2)}(E)=R_{T_1}(k_1,k_2)\phi_{(k_1,k_2+\pi)}(E),\\ T_2:\quad &\tilde{\phi}_{(k_1,k_2)}(E)=R_{T_2}(k_1,k_2)\phi_{(k_1,k_2)}(E).\end{aligned}$$

$\tilde{\phi}$ and ϕ are the basis after and before the symmetry operations.

In such a set of basis the Dirac Hamiltonian obtained by expanding the $U(1)$ SL-$[0,\pi]$ mean-field ansatz (C4) around the two cones at $\pm\mathbf{Q}$ is

$$H_{\mathrm{Dirac}}=\sum_{\mathbf{k}}\frac{\chi}{\sqrt{2}}\Psi^\dagger_{\mathbf{k}}\mu^0\sigma^3(-k_x\nu^1+k_y\nu^2)\Psi_{\mathbf{k}} \tag{C23}$$

$\mathbf{k}$ should be understood as small momenta measured from $\pm\mathbf{Q}$. Possible mass terms are $\mu^{0,1,2,3}\sigma^{1,2}\nu^0$ and $\mu^{0,1,2,3}\sigma^{0,3}\nu^3$. However not all of them are allowed by symmetry. Here we numerate all symmetry operations and associated operator transformations:

Spin rotation along $\hat{z}$-axis by angle θ:

$$\Psi^\dagger_{\mathbf{k}}\to\Psi^\dagger_{\mathbf{k}}e^{\mathrm{i}\frac{\theta}{2}}$$

Spin rotation along $\hat{y}$-axis by π:

$$\Psi^\dagger_{\mathbf{k}}\to\Psi^T_{-\mathbf{k}}\mu^2\sigma^2\nu^2$$

Time reversal $\boldsymbol{T}$:

$$\Psi^\dagger_{\mathbf{k}}\to\Psi^\dagger_{\mathbf{k}}(-\mathrm{i}\sigma^2)$$

Translation T_1:

$$\Psi^\dagger_{\mathbf{k}}\to\Psi^\dagger_{\mathbf{k}}(-\mu^3)$$

Translation T_2:

$$\Psi^\dagger_{\mathbf{k}}\to\Psi^\dagger_{\mathbf{k}}(-\mathrm{i}\mu^3)$$

Considering the above conditions, the only symmetry-allowed mass terms are $\sum_{\mathbf{k}}\Psi^\dagger_{\mathbf{k}}m_{1,2}\Psi_{\mathbf{k}}$ with $m_1=\mu^0\sigma^1\nu^0$ and $m_2=\mu^3\sigma^3\nu^3$.

The transformation rules for mirror reflection $\boldsymbol{\sigma}$ and $\pi/3$ rotation C_6 depend on the choice of $g_{\boldsymbol{\sigma}},g_{C_6}$ in the PSG. In general we have

$$\begin{aligned}\boldsymbol{\sigma}:\quad &\Psi^\dagger_{\mathbf{k}}\to\Psi^\dagger_{\boldsymbol{\sigma}\mathbf{k}}M_{\boldsymbol{\sigma}}(g_{\boldsymbol{\sigma}}),\\ C_6:\quad &\Psi^\dagger_{\mathbf{k}}\to\Psi^\dagger_{C_6\mathbf{k}}M_{C_6}(g_{C_6}).\end{aligned}$$

Using the basis (C22) the 8×8 matrices $M_{\boldsymbol{\sigma},C_6}$ can be expressed in terms of Pauli matrices $\boldsymbol{\mu}\otimes\boldsymbol{\sigma}\otimes\boldsymbol{\nu}$. For the four Z_2 spin liquid we have

$$\begin{aligned}M_{\boldsymbol{\sigma}}(g_{\boldsymbol{\sigma}}=\tau^0)&=\mu^3\otimes\sigma^0\otimes\begin{pmatrix}0&e^{-\mathrm{i}\frac{1}{12}\pi}\\e^{-\mathrm{i}\frac{5}{12}\pi}&0\end{pmatrix},\\ M_{\boldsymbol{\sigma}}(g_{\boldsymbol{\sigma}}=\mathrm{i}\tau^3)&=\mu^3\otimes\sigma^3\otimes\begin{pmatrix}0&e^{\mathrm{i}\frac{5}{12}\pi}\\e^{\mathrm{i}\frac{1}{12}\pi}&0\end{pmatrix};\\ M_{C_6}(g_{C_6}=\tau^0)&=\begin{pmatrix}1&0\\0&\mathrm{i}\end{pmatrix}\otimes\sigma^0\otimes e^{\mathrm{i}\frac{7}{6}\pi\nu^3},\\ M_{C_6}(g_{C_6}=\mathrm{i}\tau^3)&=\begin{pmatrix}\mathrm{i}&0\\0&-1\end{pmatrix}\otimes\sigma^0\otimes e^{\mathrm{i}\frac{1}{6}\pi\nu^3}.\end{aligned}$$

It turns out in $Z_2[0,\pi]\beta$ state, only the 1st mass term $m_1=\mu^0\sigma^1\nu^0$ is invariant under $\boldsymbol{\sigma}$ and C_6 operations. In other 3 states neither mass terms $m_{1,2}$ are symmetry-allowed. As a result we only have one gapped Z_2 spin liquid, *i.e.* $Z_2[0,\pi]\beta$ state in the neighborhood of $U(1)$ Dirac SL-$[0,\pi]$ state.

Let's consider mean-field bonds up to 2nd neighbor for ansatz $Z_2[0,\pi]\beta$. Perturbations to the two Dirac cones of $U(1)$ SL-$[0,\pi]$ with $\lambda_3=(\sqrt{3}-1)a_0$ in general has the following form

$$\begin{aligned}\delta H_0=&\big[\lambda_3-(\sqrt{3}-1)a_0-(\sqrt{3}+1)b_0\big]\mu^0\sigma^3\nu^0\\ &+\big[(\sqrt{3}+1)b_1-\lambda_2-(\sqrt{3}-1)a_1\big]\mu^0\sigma^1\nu^0\end{aligned} \tag{C24}$$

This means we need either 1st neighbor (a_1) or 2nd neighbor (b_1) pairing term to open up a gap in the spectrum. Meanwhile these pairing terms break the original $U(1)$ symmetry down to Z_2 symmetry.

Appendix D: Z_2 spin liquids in the neighborhood of uniform RVB state

The mean-field ansatz of the uniform RVB state is simple:

$$H_{MF} = \chi \sum_{<ij>,\sigma} f^\dagger_{i,\sigma} f_{j,\sigma} \tag{D1}$$

where χ is a real parameter and $< ij >$ represents sites i, j being nearest neighbor (n.n.) of each other. It's straightforward to show the PSG elements of such a mean-field ansatz are

$$G_U(x, y, s) = g_U, \quad U = T_{1,2},\ \boldsymbol{T},\ \boldsymbol{\sigma},\ C_6. \tag{D2}$$

and $SU(2)$ matrices g_U satisfy

$$g_{\boldsymbol{T}} \tau^3 g^\dagger_{\boldsymbol{T}} = -\tau^3, \qquad g_U \tau^3 g^\dagger_U = \tau^3, \quad U = T_{1,2}, \boldsymbol{\sigma}, C_6. \tag{D3}$$

It turns out there are only 4 gauge-inequivalent Z_2 PSGs as solutions to (A4)-(A16) with the form (D2). In other words, there are only 4 different Z_2 in the neighborhood of a uniform RVB states. Choosing a proper gauge they all satisfy $g_{\boldsymbol{T}} = \mathrm{i}\tau^1$, $g_{T_{1,2}} = \tau^0$ and $\eta_{T_{1,2}\boldsymbol{T}} = \eta_{12} = \eta_{C_6 T_{1,2}} = \eta_{\boldsymbol{\sigma} T_{1,2}} = 1$, $\eta_{\boldsymbol{T}} = -1$. These four states are characterized by:

$$(\#1)\ Z_2[0,0]A: \quad g_{\boldsymbol{\sigma}} = g_{C_6} = \tau^0, \qquad \eta_{\boldsymbol{\sigma T}} = \eta_{C_6 \boldsymbol{T}} = \eta_{\boldsymbol{\sigma}} = \eta_{C_6} = \eta_{\boldsymbol{\sigma} C_6} = 1. \tag{D4}$$

$$(\#5)\ Z_2[0,0]B: \quad g_{\boldsymbol{\sigma}} = \tau^0,\ g_{C_6} = \mathrm{i}\tau^3, \qquad \eta_{\boldsymbol{\sigma T}} = \eta_{\boldsymbol{\sigma}} = 1,\ \eta_{C_6 \boldsymbol{T}} = \eta_{C_6} = \eta_{\boldsymbol{\sigma} C_6} = -1. \tag{D5}$$

$$(\#15)\ Z_2[0,0]C: \quad g_{\boldsymbol{\sigma}} = \mathrm{i}\tau^3,\ g_{C_6} = \tau^0, \qquad \eta_{\boldsymbol{\sigma T}} = \eta_{\boldsymbol{\sigma}} = -1,\ \eta_{C_6 \boldsymbol{T}} = \eta_{C_6} = \eta_{\boldsymbol{\sigma} C_6} = 1. \tag{D6}$$

$$(\#13)\ Z_2[0,0]D: \quad g_{\boldsymbol{\sigma}} = g_{C_6} = \mathrm{i}\tau^3, \qquad \eta_{\boldsymbol{\sigma T}} = \eta_{C_6 \boldsymbol{T}} = \eta_{\boldsymbol{\sigma}} = \eta_{C_6} = \eta_{\boldsymbol{\sigma} C_6} = -1. \tag{D7}$$

It turns out these four Z_2 SLs around uniform RVB state are all gapped as shown in TABLE II.

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